MATH 3060 Assignment 3 solution

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1. We will show that the complement of H is open, see question 3b here for a direct proof that H is closed. Let $x \notin H$, then we can find $i \in \mathbb{N}$ so that $\delta := |x_i| - 1/i > 0$. Then for any $y \in B_\delta(x)$, we have

$$|y_i| \geq |x_i| - |y_i - x_i| \geq |x_i| - d_2(x, y) > |x_i| - \delta = 1/i.$$

   Hence $B_\delta(x) \subset H^c$. This shows H^c is open and thus H is closed.

2. In this question, we will use the following result of lecture 8. For a closed subset A of a metric space X, and $x \in X$. Denote

$$d(x, A) = \inf_{y \in A} d(x, y)$$

   This is a continuous function in x and $d(x, A) = 0$ if and only if $x \in A$.

   (a) Define

$$f(x) := 1 - \frac{d(x, A)}{d(x_0, A)}$$

   (b) Define

$$g(x) := \frac{d(x, A)}{d(x, A) + d(x, B)}$$

3. (a) The closure is $[0, 1]$: For one thing, $[0, 1]$ is a closed set containing $\cup_{k=1}^{\infty}(\frac{1}{k+1}, \frac{1}{k})$; for another, let $x \in [0, 1]$, and $\delta > 0$, we want to show $B_\delta(x)$ has nonempty intersection with some $(\frac{1}{k+1}, \frac{1}{k})$. If $x \in \cup_{k=1}^{\infty}(\frac{1}{k+1}, \frac{1}{k})$, then we are done, otherwise $x = 1/k$ or $x = 0$. If $x = 0$, then $(\frac{1}{k+1}, \frac{1}{k}) \subset B_\delta(x)$ for $k > 1/\delta$, if $x = 1/k$, then $B_{\delta(x)}$ has nonempty intersection with $(\frac{1}{k+1}, \frac{1}{k})$.
   Since $\cup_{k=1}^{\infty}(\frac{1}{k+1}, \frac{1}{k})$ is open, it is the interior of itself.
   Boundary is equal to the intersection of the closure and the complement of interior:

$$\partial\left(\cup_{k=1}^{\infty}(\frac{1}{k+1}, \frac{1}{k})\right) = [0, 1] \cap \left(\cup_{k=1}^{\infty}(\frac{1}{k+1}, \frac{1}{k})\right)^c = \{0\} \cup \{\frac{1}{k}, k \in \mathbb{N}\}.$$

(b) The closure is the whole $\mathbb{R}^2$, the interior is $\mathbb{R}^2 \setminus (\{(0,0)\} \cup \{(1/n, 0) : n \in \mathbb{N}\})$. The boundary is $\{(0,0)\} \cup \{(1/n,0) : n \in \mathbb{N}\}$. Reasons are similar to part (a).

(c) Let $F = \{f \in C[0,1] : f(0) = f(1)\}$.
F is closed, because if $f_n \in F$, and $f_n \to f$, we have

$$\begin{aligned}|f(0) - f(1)| &\leq |f(0) - f_n(0)| + |f(1) - f_n(1)| + |f_n(0) - f_n(1)| \\ &\leq 2d_\infty(f_n, f) \to 0,\end{aligned}$$

i.e. $f \in F$.
F has no interior point, for if $f \in F$, and $\epsilon > 0$, then $d_\infty(f, f + \epsilon x^2/2) = \epsilon/2 < \epsilon$, while $f(0) + \epsilon \cdot 0^2/2 \neq f(1) + \epsilon \cdot 1^2/2$.
The boundary is $F \cap \varnothing^c = F$.

4. We prove the statement by induction, the case $n = 2$ is the usual Hölder's inequality. For the induction step, note

$$\int |f_1 f_2 \cdots f_n| \leq ||f_1||_{p_1} ||f_2 \cdots f_n||_q$$

where $1/p_1 + 1/q = 1$. It suffices to show $||f_2 \cdots f_n||_q \leq ||f_2||_{p_2} \cdots ||f_n||_{p_n}$. This is equivalent to

$$\begin{aligned} &\int |f_2^q \cdots f_n^q| \leq ||f_2||_{p_2}^q \cdots ||f_n||_{p_n}^q \\ \iff &\int |f_2^q \cdots f_n^q| \leq ||f_2^q||_{\frac{p_2}{q}} \cdots ||f_n^q||_{\frac{p_n}{q}}. \end{aligned}$$

The last line follows from the induction hypothesis, because

$$\frac{1}{p_2/q} + \cdots + \frac{1}{p_n/q} = 1$$